8-4

1. Plan

Objectives

1 To use the properties of logarithms

Examples

1 Identifying the Properties of Logarithms
2 Simplifying Logarithms
3 Expanding Logarithms
4 Real-World Connection

Professional Development

Math Background

Many students want to believe that $\log_b(M + N) = \log_b M + \log_b N$. However, this would be equivalent to adding the exponents in expressions such as $x^2 + x^3$, which cannot be done. Thus, it is reasonable that there is *no* property for the logarithm of a sum. Do not confuse this with the valid property for a logarithm of a product, $\log_b(MN) = \log_b M + \log_b N$.

More Math Background: p. 428C

Lesson Planning and Resources

See p. 428E for a list of the resources that support this lesson.

PowerPoint

Bell Ringer Practice

Check Skills You'll Need
For intervention, direct students to:

Logarithmic Functions as Inverses
Lesson 8-3: Example 3
Extra Skills and Word Problems Practice, Ch. 8

Algebraic Expressions
Lesson 1-2: Examples 1, 2
Extra Skills and Word Problems Practice, Ch. 1

8-4 Properties of Logarithms

What You'll Learn

- To use the properties of logarithms

. . . And Why

To relate sound intensity and decibel level, as in Example 4

Check Skills You'll Need — GO for Help Lessons 8-3 and 1-2

Simplify each expression.

1. $\log_2 4 + \log_2 8$ **5**
2. $\log_3 9 - \log_3 27$ **−1**
3. $\log_2 16 \div \log_2 64$ **$\frac{2}{3}$**

Evaluate each expression for $x = 3$.

4. $x^3 - x$ **24**
5. $x^5 \cdot x^2$ **2187**
6. $\frac{x^6}{x^9}$ **$\frac{1}{27}$**
7. $x^3 + x^2$ **36**

1 Using the Properties of Logarithms

1. 0, 0.301, 0.477, 0.602, 0.699, 0.778, 0.845, 0.903, 0.954, 1, 1.176, 1.301

2. The sum of the logarithms equals the log of the product.

Activity: Properties of Logarithms

1. Complete the table. Round to the nearest thousandth.

x	1	2	3	4	5	6	7	8	9	10	15	20
log *x*	■	■	■	■	■	■	■	■	■	■	■	■

2. Complete each pair of statements. What do you notice? **See left.**
   a. log 3 + log 5 = ■ and log (3 · 5) = ■ **1.176, 1.176**
   b. log 1 + log 7 = ■ and log (1 · 7) = ■ **0.845, 0.845**
   c. log 2 + log 4 = ■ and log (2 · 4) = ■ **0.903, 0.903**
   d. log 10 + log 2 = ■ and log (10 · 2) = ■ **1.301, 1.301**
3. Complete the statement: log *M* + log *N* = ■. **log (*MN*)**
4. a. **Make a Conjecture** How could you rewrite the expression $\log \frac{M}{N}$ using the expressions log *M* and log *N*? **$\log \frac{M}{N} = \log M - \log N$**
   b. Use your calculator to verify your conjecture for several values of *M* and *N*. **Check students' work.**

The properties of logarithms are summarized below.

Key Concepts

Properties — **Properties of Logarithms**

For any positive numbers, M, N, and b, $b \neq 1$,

$\log_b MN = \log_b M + \log_b N$ **Product Property**

$\log_b \frac{M}{N} = \log_b M - \log_b N$ **Quotient Property**

$\log_b M^x = x \log_b M$ **Power Property**

Differentiated Instruction Solutions for All Learners

Special Needs L1
Students who wear hearing aids, or who have difficulty hearing, may be sensitive about a discussion of noise levels and decibels. However, if these students are comfortable with the topic, invite them to contribute some of their special knowledge.
learning style: verbal

Below Level L2
Have students state the Properties of Logarithms in words. Give illustrations using those words. Discuss whether these agree with the Property chosen.
learning style: verbal

You can use the properties of logarithms to rewrite logarithmic expressions.

1 EXAMPLE Identifying the Properties of Logarithms

State the property or properties used to rewrite each expression.

a. $\log_2 8 - \log_2 4 = \log_2 2$

Quotient Property: $\log_2 8 - \log_2 4 = \log_2 \frac{8}{4} = \log_2 2$

b. $\log_b x^3y = 3\log_b x + \log_b y$

Product Property: $\log_b x^3y = \log_b x^3 + \log_b y$

Power Property: $\log_b x^3 + \log_b y = 3\log_b x + \log_b y$

Quick Check 1 State the property or properties used to rewrite each expression.

a. $\log_5 2 + \log_5 6 = \log_5 12$ **Product Property**

b. $3\log_b 4 - 3\log_b 2 = \log_b 8$ **Power Property, Quotient Property**

You can write the sum or difference of logarithms with the same base as a single logarithm.

2 EXAMPLE Simplifying Logarithms

GO Online

Video Tutor Help
Visit: PHSchool.com
Web Code: age-0775

Write each logarithmic expression as a single logarithm.

a. $\log_3 20 - \log_3 4$

$\log_3 20 - \log_3 4 = \log_3 \frac{20}{4}$ **Quotient Property**

$= \log_3 5$ **Simplify.**

b. $3\log_2 x + \log_2 y$

$3\log_2 x + \log_2 y = \log_2 x^3 + \log_2 y$ **Power Property**

$= \log_2 (x^3y)$ **Product Property**

So $\log_3 20 - \log_3 4 = \log_3 5$, and $3\log_2 x + \log_2 y = \log_2 (x^3y)$.

Quick Check 2 **a.** Write $3\log 2 + \log 4 - \log 16$ as a single logarithm. **log 2**

b. **Critical Thinking** Can you write $3\log_2 9 - \log_6 9$ as a single logarithm? Explain. **No; they do not have the same base.**

3a. $\log_2 7 + \log_2 b$

b. $2\log y - 2\log 3$

c. $3\log_7 a + 4\log_7 b$

You can sometimes write a single logarithm as a sum or difference of two or more logarithms.

3 EXAMPLE Expanding Logarithms

Vocabulary Tip

In mathematics, to expand means "to show the full form of."

Expand each logarithm.

a. $\log_5 \frac{x}{y}$

$= \log_5 x - \log_5 y$ **Quotient Property**

b. $\log 3r^4$

$= \log 3 + \log r^4$ **Product Property**

$= \log 3 + 4\log r$ **Power Property**

Quick Check 3 Expand each logarithm. **See above left.**

a. $\log_2 7b$ **b.** $\log \left(\frac{y}{3}\right)^2$ **c.** $\log_7 a^3b^4$

2. Teach

Guided Instruction

Activity

Teaching Tip

Remind students that a logarithm is an exponent. Therefore, it seems reasonable that there are properties for operations with logarithms that are similar to the properties for operations with exponents.

2 EXAMPLE Math Tip

Point out to students that the bases are the same within each expression in part (a) and within each expression in part (b). The properties for logarithms do not apply unless the bases are the same.

PowerPoint

Additional Examples

1 State the property or properties used to rewrite each expression.

a. $\log 6 = \log 2 + \log 3$ **Product Property**

b. $\log_b \frac{x^2}{y} = 2\log_b x - \log_b y$ **Quotient Property and Power Property**

2 Write each expression as a single logarithm.

a. $\log_4 64 - \log_4 16$ **$\log_4 4$ or 1**

b. $6\log_5 x + \log_5 y$ **$\log_5(x^6y)$**

Advanced Learners L4

Ask students to explain how the properties of Logarithms on page 454 are similar to and how they are different from the properties of exponents.

learning style: verbal

English Language Learners ELL

Emphasize the correct reading of a logarithmic expression. For $\log_b 2$, students should say "log base *b* of 2." Also, clarify that the term *log* by itself is meaningless. You cannot say $y = \log$ just as you cannot say $y = \sqrt{\ }$.

learning style: verbal

3 EXAMPLE **Error Prevention**

Discuss with students the fact that writing their exponents and bases clearly will help them avoid errors. It is easy to misread and confuse these smaller digits so that $\log_5 \frac{x}{y}$ might be misread as $\log 5\left(\frac{x}{y}\right)$.

PowerPoint

Additional Examples

3 Expand each logarithm.

a. $\log_7 \frac{t}{u}$ $\log_7 t - \log_7 u$

b. $\log 4p^3$ $\log 4 + 3 \log p$

4 Manufacturers of a vacuum cleaner want to reduce its sound intensity to 40% of the original intensity. By how many decibels would the loudness be reduced? **about four decibels.**

Resources

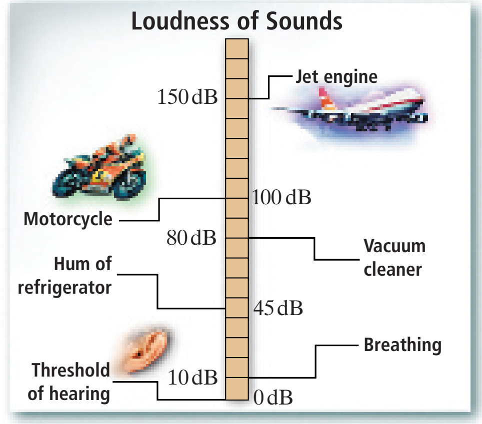
- Daily Notetaking Guide 8-4 L3
- Daily Notetaking Guide 8-4—Adapted Instruction L1

Closure

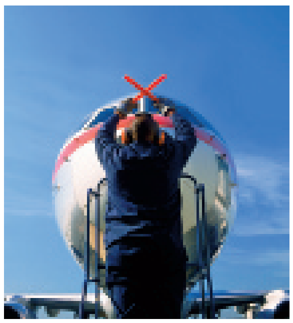
Ask students to write a paragraph describing how to use one logarithm property to simplify logarithm expressions. You may wish to assign properties so that all three are covered.

Logarithms are used to model sound. The intensity of a sound is a measure of the energy carried by the sound wave. The greater the intensity of a sound, the louder it seems. This apparent loudness L is measured in decibels.

You can use the formula $L = 10 \log \frac{I}{I_0}$, where I is the intensity of the sound in watts per square meter (W/m^2). I_0 is the lowest-intensity sound that the average human ear can detect.



4 EXAMPLE **Real-World Connection**



Real-World Connection

The workers who direct planes at airports must wear ear protection.

Noise Control A shipping company has started flying cargo planes out of the city airport. Residents in a nearby neighborhood have complained that the cargo planes are too loud. Suppose the shipping company hires you to design a way to reduce the intensity of the sound by half. By how many decibels would the loudness of the sound be decreased?

Relate The reduced intensity is one half of the present intensity.

Define Let I_1 = present intensity.
Let I_2 = reduced intensity.
Let L_1 = present loudness.
Let L_2 = reduced loudness.

Write $I_2 = 0.5\, I_1$

$$L_1 = 10 \log \frac{I_1}{I_0}$$

$$L_2 = 10 \log \frac{I_2}{I_0}$$

$$\begin{aligned}
L_1 - L_2 &= 10 \log \frac{I_1}{I_0} - 10 \log \frac{I_2}{I_0} && \text{Find the decrease in loudness } L_1 - L_2.\\
&= 10 \log \frac{I_1}{I_0} - 10 \log \frac{0.5 I_1}{I_0} && \text{Substitute } I_2 = 0.5 I_1.\\
&= 10 \log \frac{I_1}{I_0} - 10 \log \left(0.5 \cdot \frac{I_1}{I_0}\right) && \\
&= 10 \log \frac{I_1}{I_0} - 10\left(\log 0.5 + \log \frac{I_1}{I_0}\right) && \text{Product Property}\\
&= 10 \log \frac{I_1}{I_0} - 10 \log 0.5 - 10 \log \frac{I_1}{I_0} && \text{Distributive Property}\\
&= -10 \log 0.5 && \text{Combine like terms.}\\
&\approx 3.0 && \text{Use a calculator.}
\end{aligned}$$

The decrease in loudness would be about three decibels.

4. about 6 decibels

Quick Check **4** Suppose the shipping company wants you to reduce the sound intensity to 25% of the original intensity. By how many decibels would the loudness be reduced?

1. Product Property
2. Quotient Property
3. Power Property
4. Power Property
5. Power Property, Quotient Property
6. Power Property
7. Power Property, Quotient Property
8. Power Property, Product Property
9. Power Property, Quotient Property
10. Power Property, Product Property

EXERCISES

For more exercises, see *Extra Skill and Word Problem Practice.*

Practice and Problem Solving

A Practice by Example

Example 1 (page 455)

GO for Help

1–10. See margin p. 456.

State the property or properties used to rewrite each expression.

1. $\log 4 + \log 5 = \log 20$
2. $\log_3 32 - \log_3 8 = \log_3 4$
3. $\log z^2 = 2\log z$
4. $\log_6 \sqrt[n]{x^p} = \frac{p}{n}\log_6 x$
5. $8\log 2 - 2\log 8 = \log 4$
6. $\log \sqrt[3]{3x} = \frac{1}{3}\log 3x$
7. $3\log_4 5 - 3\log_4 3 = \log_4\left(\frac{5}{3}\right)^3$
8. $2\log w + 4\log z = \log w^2z^4$
9. $2\log_2 m - 4\log_2 n = \log_2 \frac{m^2}{n^4}$
10. $\log_b \frac{1}{8} + 3\log_b 4 = \log_b 8$

Example 2 (page 455)

Write each logarithmic expression as a single logarithm.

11. $\log 7 + \log 2$ **$\log 14$**
12. $\log_2 9 - \log_2 3$ **$\log_2 3$**
13. $5\log 3 + \log 4$ **$\log 972$**
14. $\log 8 - 2\log 6 + \log 3$ **$\log \frac{2}{3}$**
15. $4\log m - \log n$ **$\log \frac{m^4}{n}$**
16. $\log 5 - k\log 2$ **$\log \frac{5}{2^k}$**
17. $\log_6 5 + \log_6 x$ **$\log_6 5x$**
18. $\log_7 x + \log_7 y - \log_7 z$ **$\log_7 \frac{xy}{z}$**

Example 3 (page 455)

Expand each logarithm. 19–30. See margin.

19. $\log x^3y^5$
20. $\log_7 22xyz$
21. $\log_4 5\sqrt{x}$
22. $\log 3m^4n^{-2}$
23. $\log_5 \frac{r}{s}$
24. $\log_3 (2x)^2$
25. $\log_3 7(2x - 3)^2$
26. $\log \frac{a^2b^3}{c^4}$
27. $\log \sqrt{\frac{2x}{y}}$
28. $\log_8 8\sqrt{3a^5}$
29. $\log \frac{s\sqrt{7}}{t^2}$
30. $\log_b \frac{1}{x}$

Example 4 (page 456)

31. One brand of ear plugs claims to block the sound of snoring as loud as 22 dB. A second brand claims to block snoring that is eight times as intense. If the claims are true, for how many more decibels is the second brand effective? **9 dB**

32. A sound barrier along a highway reduced the intensity of the noise reaching a community by 95%. By how many decibels was the noise reduced? **13 dB**

B Apply Your Skills

Use the properties of logarithms to evaluate each expression.

33. $\log_2 4 - \log_2 16$ **-2**
34. $3\log_2 2 - \log_2 4$ **1**
35. $\log_3 3 + 5\log_3 3$ **6**
36. $\log 1 + \log 100$ **2**
37. $\log_6 4 + \log_6 9$ **2**
38. $2\log_8 4 - \frac{1}{3}\log_8 8$ **1**
39. $2\log_3 3 - \log_3 3$ **1**
40. $\frac{1}{2}\log_5 1 - 2\log_5 5$ **-2**
41. $\log_9 \frac{1}{3} + 3\log_9 3$ **1**

42. **Error Analysis** Explain why the expansion below of $\log_4 \sqrt{\frac{t}{s}}$ is incorrect. Then do the expansion correctly. **See left.**

$$\log_4\sqrt{\frac{t}{s}} = \frac{1}{2}\log_4 \frac{t}{s}$$
$$= \frac{1}{2}\log_4 t - \log_4 s$$

42. The coefficient $\frac{1}{2}$ is missing in $\log_4 s$; $\log_4 \sqrt{\frac{t}{s}} = \frac{1}{2}\log_4 \frac{t}{s} = \frac{1}{2}(\log_4 t - \log_4 s) = \frac{1}{2}\log_4 t - \frac{1}{2}\log_4 s.$

43. **Open-Ended** Write log 150 as a sum or difference of two logarithms. **Answers may vary. Sample: log 150 = log 15 + log 10.**

3. Practice

Assignment Guide

1 A B	1-87
C Challenge	88-90
Test Prep	91-95
Mixed Review	96-108

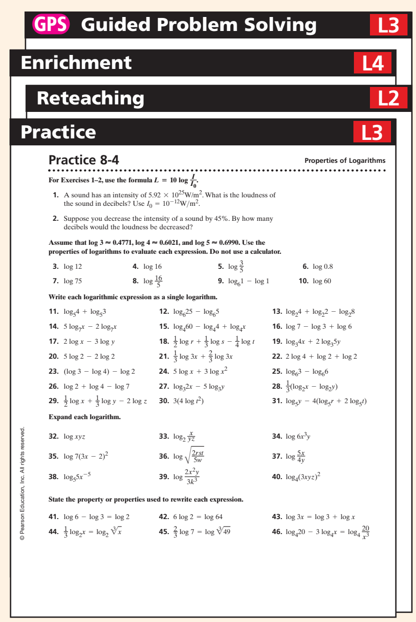
Homework Quick Check

To check students' understanding of key skills and concepts, go over Exercises 25, 32, 56, 57, 72, 75.

Diversity

Exercise 32 Some students may not know what a sound barrier along a highway looks like. Ask students to bring pictures, draw a sketch, or tell where one can be seen nearby. Discuss the fact that highway sound barriers are often built to prevent highway noise from affecting neighborhoods.

Differentiated Instruction Resources



19. $3\log x + 5\log y$
20. $\log_7 22 + \log_7 x + \log_7 y + \log_7 z$
21. $\log_4 5 + \frac{1}{2}\log_4 x$
22. $\log 3 + 4\log m - 2\log n$
23. $\log_5 r - \log_5 s$
24. $2\log_3 2 + 2\log_3 x$
25. $\log_3 7 + 2\log(2x - 3)$
26. $2\log a + 3\log b - 4\log c$
27. $\frac{1}{2}\log 2 + \frac{1}{2}\log x - \frac{1}{2}\log y$
28. $1 + \frac{1}{2}\log_8 3 + \frac{5}{2}\log_8 a$
29. $\log s + \frac{1}{2}\log 7 - 2\log t$
30. $-\log_b x$

4. Assess & Reteach

PowerPoint

Lesson Quiz

Write each expression as a single logarithm. State the property you used.

1. $\log 12 - \log 3$ **$\log 4$; Quotient Property**
2. $3 \log_{11} 5 + \log_{11} 7$ **$\log_{11}(5^3 \cdot 7)$; Power Property and Product Property**

Expand each logarithm.

3. $\log_c \frac{a}{b}$ **$\log_c a - \log_c b$**
4. $\log_3 x^4$ **$4 \log_3 x$**

Use the properties of logarithms to evaluate each expression.

5. $\log 0.001 + \log 100$ **-1**
6. $\frac{1}{2} \log_y y$ **$\frac{1}{2}$**

Alternative Assessment

Have students work in small groups to prepare an informal proof of an expanded logarithm from Exercises 79–87. They should justify each step, using the Properties of Logarithms. Then ask each group to present their proof at the board, and encourage class discussion of the proof.

58. True; $\log_2 4 = 2$ and $\log_2 8 = 3$.

59. False; $\frac{1}{2} \log_3 3 = \log_3 3^{\frac{1}{2}}$, not $\log_3 \frac{3}{2}$.

60. True; it is an example of the Power Property since $8 = 2^3$.

61. False; the two logs have different bases.

Assume that $\log 4 \approx 0.6021$, $\log 5 \approx 0.6990$, and $\log 6 \approx 0.7782$. Use the properties of logarithms to evaluate each expression. Do not use your calculator.

44. $\log 24$ **1.3803** **45.** $\log 30$ **1.4772** **46.** $\log 16$ **1.2042**

47. $\log 125$ **2.097** **48.** $\log 1.5$ **0.1761** **49.** $\log 0.8$ **-0.0969**

50. $\log \frac{1}{4}$ **-0.6021** **51.** $\log \frac{1}{25}$ **-1.398** **52.** $\log 25$ **1.398**

53. $\log \frac{1}{6}$ **-0.7782** **54.** $\log 36$ **1.5564** **55.** $\log \sqrt{5}$ **0.3495**

Real-World Connection

Decibel meters are used to measure sound levels.

56. Noise Control New components reduce the sound intensity of a certain model of vacuum cleaner from 10^{-4} W/m^2 to 6.31×10^{-6} W/m^2. By how many decibels do these new components reduce the vacuum cleaner's loudness? **12 dB**

GPS

57. Reasoning If $\log x = 5$, what is the value of $\frac{1}{x}$? **0.00001**

Write *true* or *false* for each statement. Justify your answer. **58–67. See margin.**

58. $\log_2 4 + \log_2 8 = 5$ **59.** $\log_3 \frac{3}{2} = \frac{1}{2} \log_3 3$

60. $\log_3 8 = 3 \log_3 2$ **61.** $\log_5 16 - \log 2 = \log_5 8$

62. $\log(x - 2) = \frac{\log x}{\log 2}$ **63.** $\frac{\log_b x}{\log_b y} = \log_b \frac{x}{y}$

64. $(\log x)^2 = \log x^2$ **65.** $\log_4 7 - \log_4 3 = \log_4 4$

66. $\log x + \log(x^2 + 2) = \log(x^3 + 2x)$ **67.** $\log_2 3 + \log_3 2 = \log_6 6$

68. $\log_2 x - 4 \log_2 y = \log_2 \frac{x}{y^4}$ **69.** $\log_b \frac{1}{8} + 3 \log_b 4 = \log_b 8$

68–69. See margin p. 459.

70. Construction Suppose you are the supervisor on a road construction job. Your team is blasting rock to make way for a roadbed. One explosion has an intensity of 1.65×10^{-2} W/m^2. What is the loudness of the sound in decibels? (Use $I_0 = 10^{-12}$ W/m^2.) **102 dB**

71. Critical Thinking Can you expand $\log_3(2x + 1)$? Explain.

71. No; the expression $(2x + 1)$ is a sum, so it is not covered by the Product, Quotient, or Power properties.

72. Writing Explain why $\log(5 \cdot 2) \neq \log 5 \cdot \log 2$. **See margin p. 459.**

Write each logarithmic expression as a single logarithm. **73–78. See left.**

73. $\frac{1}{4} \log_3 2 + \frac{1}{4} \log_3 x$ **74.** $\frac{1}{2}(\log_x 4 + \log_x y) - 3 \log_x z$

75. $2 \log 3 - \frac{1}{2} \log 4 + \frac{1}{2} \log 9$ **76.** $x \log_4 m + \frac{1}{y} \log_4 n - \log_4 p$

77. $\left(\frac{2 \log_b x}{3} + \frac{3 \log_b y}{4}\right) - 5 \log_b z$ **78.** $\frac{\log z - \log 3}{4} - 5 \frac{\log x}{2}$

73. $\log_3 \sqrt[4]{2x}$

74. $\log_x \frac{2\sqrt{y}}{z^3}$

75. $\log \frac{27}{2}$

76. $\log_4 \frac{m^x n^{\frac{1}{y}}}{p}$

77. $\log_b \frac{\sqrt[3]{x^2}\sqrt[4]{y^3}}{z^5}$

78. $\log \frac{\sqrt[4]{z}}{\sqrt[4]{3}\sqrt{x^5}}$

Expand each logarithm. **79–87. See back of book.**

79. $\log\left(\frac{2\sqrt{x}}{5}\right)^3$ **80.** $\log \frac{m^3}{n^4 p^{-2}}$ **81.** $\log 2\sqrt{\frac{4r}{s^2}}$

82. $\log_b \frac{\sqrt{x}\sqrt[3]{y^2}}{\sqrt[5]{z^2}}$ **83.** $\log_4 \frac{\sqrt{x^5 y^7}}{zw^4}$ **84.** $\log \frac{\sqrt{x^2 - 4}}{(x + 3)^2}$

85. $\log \sqrt{\frac{x\sqrt{2}}{y^2}}$ **86.** $\log_3\left[(xy)^{\frac{1}{3}} \div z^2\right]^3$ **87.** $\log_7 \frac{\sqrt{r + 9}}{s^2 t^{\frac{1}{3}}}$

GO Online Homework Help

Visit: PHSchool.com
Web Code: age-0804

62. False; this is not an example of the Quotient Property. $\log(x - 2) \neq \log x - \log 2$.

63. False; $\log_b \frac{x}{y} = \log_b x - \log_b y$.

64. False; the exponent on the left means $\log x$, quantity squared, not the log of x^2.

65. False; $\log_4 7 - \log_4 3 = \log_4 \frac{7}{3}$, not $\log_4 4$.

66. True; $\log x + \log(x^2 + 2) = \log x(x^2 + 2)$, which equals $\log(x^3 + 2x)$.

67. False; the three logs have different bases.

Challenge

88. Let $u = \log_b M$, and let $v = \log_b N$. Prove the Product Property of Logarithms by completing the equations below.

Statement	Reason
$u = \log_b M$	Given
$b^u = M$	Rewrite in exponential form.
$v = \blacksquare$	Given
$b^v = \blacksquare$	Rewrite in exponential form.
$MN = b^u b^v = b^{\blacksquare}$	Apply the Product Property of Exponents.
$\log_b MN = \blacksquare$	Take the logarithm of each side.
$\log_b MN = \log_b \blacksquare + \log_b \blacksquare$	Substitute $\log_b M$ for u and $\log_b N$ for v.

88. $v = \log_b N$
$b^v = N$
$MN = b^u \cdot b^v = b^{u+v}$
$\log_b MN = u + v$
$\log_b MN = \log_b M + \log_b N$

89. Let $u = \log_b M$. Prove the Power Property of logarithms. **See margin.**

90. Let $u = \log_b M$ and $v = \log_b N$. Prove the Quotient Property of logarithms. **See margin.**

Test Prep

Multiple Choice

91. Which statement is NOT correct? **B**
A. $\log_2 25 = 2 \cdot \log_2 5$
B. $\log_3 16 = 2 \cdot \log_3 8$
C. $\log_5 27 = 3 \cdot \log_5 3$
D. $\log_8 10{,}000 = 4 \cdot \log_8 10$

92. Which expression is equal to $\log_7 5 + \log_7 3$? **G**
F. $\log_7 8$
G. $\log_7 15$
H. $\log_7 125$
J. $\log_{49} 15$

93. Which expression is equal to $\log_5 x + 4 \cdot \log_5 y - 2 \cdot \log_5 z$? **D**
A. $\log_5 (-8xyz)$
B. $-\log_5 \frac{4xy}{2z}$
C. $\log_5 \frac{(xy)^4}{z^2}$
D. $\log_5 \frac{xy^4}{z^2}$

Short Response

94. $\log_5 10 \approx 1.4307$ and $\log_5 20 \approx 1.8614$. Find the value of $\log_5 \left(\frac{1}{2}\right)$ without using a calculator. Explain how you found the value. **See margin.**

Extended Response

95. Use the properties of logarithms to write log 12 in four different ways. Name each property you use. **See back of book.**

Mixed Review

GO for Help

Lesson 8-3 Write each equation in logarithmic form.

96. $49 = 7^2$ **$\log_7 49 = 2$**
97. $5^3 = 125$ **$3 = \log_5 125$**
98. $\frac{1}{4} = 8^{-\frac{2}{3}}$ **$\log_8 \frac{1}{4} = -\frac{2}{3}$**
99. $5^{-3} = \frac{1}{125}$ **$-3 = \log_5 \frac{1}{125}$**

Lesson 7-5 Solve each equation. Check for extraneous solutions.

100. $\sqrt[3]{y^4} = 16$ **8, −8**
101. $\sqrt[3]{7x} - 4 = 0$ **$\frac{64}{7}$**
102. $2\sqrt{w-1} = \sqrt{w+2}$ **2**

Lesson 6-5 A polynomial equation with integer coefficients has the given roots. What additional roots can you identify?

103. $\sqrt{3}, -\sqrt{5}$ **$-\sqrt{3}, \sqrt{5}$**
104. $-i, 4i$ **$i, -4i$**
105. $2i, -4 + i$ **$-2i, -4 - i$**
106. $\sqrt{2}, i - 1$ **$-\sqrt{2}, -i - 1$**
107. $-\sqrt{7}, -\sqrt{11}$ **$\sqrt{7}, \sqrt{11}$**
108. $-2i + 3, i$ **$2i + 3, -i$**

Test Prep

Resources
For additional practice with a variety of test item formats:
- Standardized Test Prep, p. 483
- Test-Taking Strategies, p. 478
- Test-Taking Strategies with Transparencies

89. 1. $u = \log_b M$ (given)
2. $b^u = M^b$ (Rewrite in exponential form.)
3. $(b^u)^x = M^x$ (Raise each side to x power.)
4. $b^{ux} = M^x$ (Power Property of exponents)
5. $\log_b b^{ux} = \log_b M^x$ (Take the log of each side.)
6. $ux = \log_b M^x$ (Simplify.)
7. $\log_b M^x = x \cdot \log_b M$ (substitution)

90. 1. $u = \log_b M$ (given)
2. $b^u = M$ (Rewrite in exponential form.)
3. $v = \log_b N$ (given)
4. $b^v = N$ (Rewrite in exponential form.)
5. $\frac{M}{N} = \frac{b^u}{b^v} = b^{u-v}$ (Quotient Property of Exponents)
6. $\log_b \frac{M}{N} = \log_b b^{u-v}$ (Take the log of each side.)
7. $\log_b \frac{M}{N} = u - v$ (Simplify.)
8. $\log_b \frac{M}{N} = \log_b M - \log_b N$ (substitution)

94. [2] By the Quotient Property, $\log_5 \left(\frac{1}{2}\right) = \log_5 \left(\frac{10}{20}\right) \approx 1.4307 - 1.8614 = -0.4307$.
[1] correct answer, without work shown

68. True; the power and quotient properties are used correctly.

69. True; the left side equals $\log_b \left(\frac{1}{8} \cdot 4^3\right)$, which equals $\log_b 8$.

72. The log of a product is equal to the sum of the logs. $\log (MN) = \log M + \log N$. So $\log (5 \cdot 2) = \log 10 = 1$, $\log 5 \cdot \log 2 \approx (0.7)(0.3) = 0.21$, which is not equal to 1.

8-5 Exponential and Logarithmic Equations

What You'll Learn

- To solve exponential equations
- To solve logarithmic equations

. . . And Why

To model animal populations, as in Example 5

Check Skills You'll Need

GO for Help Lessons 8-3 and 7-4

Evaluate each logarithm.

1. $\log_9 81 \cdot \log_9 3$ **1** **2.** $\log 10 \cdot \log_3 9$ **2** **3.** $\log_2 16 \div \log_2 8$ $\frac{4}{3}$

4. Simplify $125^{-\frac{2}{3}}$. $\frac{1}{25}$

New Vocabulary

- exponential equation
- logarithmic equation
- Change of Base Formula

1 Solving Exponential Equations

An equation of the form $b^{cx} = a$, where the exponent includes a variable, is an **exponential equation.** If m and n are positive and $m = n$, then $\log\ m = \log\ n$. You can therefore solve an exponential equation by taking the logarithm of each side of the equation.

1 EXAMPLE Solving an Exponential Equation

Solve $7^{3x} = 20$.

$7^{3x} = 20$

$\log 7^{3x} = \log 20$ **Take the common logarithm of each side.**

$3x \log 7 = \log 20$ **Use the power property of logarithms.**

$x = \frac{\log 20}{3 \log 7}$ **Divide each side by 3 log 7.**

≈ 0.5132 **Use a calculator.**

Check $7^{3x} = 20$

$7^{3(0.5132)} \approx 20.00382 \approx 20$ ✓

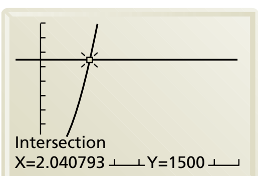
Quick Check **1** Solve each equation. Round to the nearest ten-thousandth. Check your answers.

a. $3^x = 4$ **1.2619** **b.** $6^{2x} = 21$ **0.8496** **c.** $3^{x+4} = 101$ **0.2009**

2 EXAMPLE Solving an Exponential Equation by Graphing

Solve $6^{2x} = 1500$.

Graph the equations $y_1 = 6^{2x}$ and $y_2 = 1500$. Find the point of intersection.



The solution is $x \approx 2.0408$.

Quick Check **2** Solve $11^{6x} = 786$ by graphing. **0.4634**

Differentiated Instruction Solutions for All Learners

Special Needs L1
Clarify for students that the goal in solving an exponential equation is the same as for any equation, to isolate the variable on one side of the equal sign. However, because the variable is an exponent, students must take the logarithm of both sides.
learning style: verbal

Below Level L2
When solving an exponential equation by taking the logarithm of both sides, the Power Property of Logarithms is used to solve for the variable. Review the Power Property.
learning style: verbal

8-5

1. Plan

Objectives

1 To solve exponential equations
2 To solve logarithmic equations

Examples

1 Solving an Exponential Equation
2 Using the Change of Base Formula
3 Solving an Exponential Equation by Changing Bases
4 Solving an Exponential Equation by Graphing
5 Real-World Connection
6 Solving a Logarithmic Equation
7 Using Logarithmic Properties to Solve an Equation

Math Background

Professional Development

The Change of Base Formula allows you to rewrite any logarithm in terms of a logarithm to any desired base.

More Math Background: p. 428D

Lesson Planning and Resources

See p. 428E for a list of the resources that support this lesson.

Bell Ringer Practice

PowerPoint

Check Skills You'll Need
For intervention, direct students to:

Logarithmic Functions as Inverses
Lesson 8-3: Example 3
Extra Skills and Word Problems Practice, Ch. 8

Rational Exponents
Lesson 7-4: Example 4
Extra Skills and Word Problems Practice, Ch. 7

2. Teach

Guided Instruction

1 EXAMPLE Math Tip

Point out that you can take the common logarithm (using base 10) of both sides of the equation no matter what base occurs in the equation. This means that you can use the feature of a calculator that finds the common logarithm.

2 EXAMPLE Error Prevention

When you enter $y = 6^{2x}$, be sure to use parentheses to enter it as 6^(2x).

4 EXAMPLE Connection to Biology

Although the mathematical model may indicate a population in the single digits, the reality of living creatures will not fit the mathematical model exactly. There is a minimum population level below which an endangered species will probably not be able to survive.

GO for Help

To review solving equations by tables, see Lesson 5-5.

3 EXAMPLE Solving an Exponential Equation by Tables

Solve the equation $2(1.5^x) = 6$ to the nearest hundredth.

Enter $y_1 = 2(1.5^x) - 6$. Use tabular zoom-in to find the sign change, as shown at the right.

X	Y1
2.7060	−.0085
2.7070	−.0061
2.7080	−.0037
2.7090	−.0012
2.7100	.0012
2.7110	.0036
2.7120	.0061

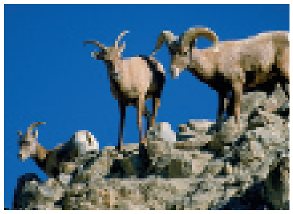
X=2.709

The solution is $x \approx 2.71$.

Quick Check

3 Solve $11^{6x} = 786$ using tables. Compare your result with your solution in Quick Check 2. **0.4634**

4 EXAMPLE Real-World Connection



Real-World Connection

The U.S. population of peninsular bighorn sheep was 1170 in 1971. By 1999, only 335 remained.

Zoology Refer to the photo. Write an exponential equation to model the decline in the population. If the decay rate remains constant, in what year might only five peninsular bighorn sheep remain in the United States?

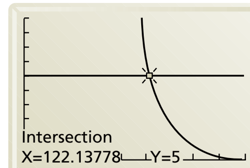
Step 1 Enter the data into your calculator. Let 0 represent the initial year, 1971.

Step 2 Use the **ExpReg** feature to find the exponential function that fits the data.

ExpReg
y = a*b^x
a = 1170
b = .9563175045

Step 3 Graph the function and the line $y = 5$.

Step 4 Find the point of intersection.



The solution is $x \approx 122$, and $1971 + 122 = 2093$, so there may be only five peninsular bighorn sheep in 2093.

Quick Check

4 The population of peninsular bighorn sheep in Mexico was approximately 6200 in 1971. By 1999, about 2300 remained. Determine the year by which only 200 peninsular bighorn sheep might remain in Mexico. **2068**

2 Solving Logarithmic Equations

To evaluate a logarithm with any base, you can use the **Change of Base Formula.**

Key Concepts

Property **Change of Base Formula**

For any positive numbers, M, b, and c, with $b \neq 1$ and $c \neq 1$,

$$\log_b M = \frac{\log_c M}{\log_c b}$$

Differentiated Instruction Solutions for All Learners

Advanced Learners L4

Have students research what form of radioactive dating is most useful for charcoal that is less than 50,000 years old.

learning style: verbal

English Language Learners ELL

In Exercise 51, have a student volunteer explain the meaning of *toxic*. Discuss the pronunciation of *protactinium*, with help from the chemistry teacher or a dictionary.

learning style: visual

5 EXAMPLE Using the Change of Base Formula

Use the Change of Base Formula to evaluate $\log_3 15$. Then convert $\log_3 15$ to a logarithm in base 2.

$\log_3 15 = \frac{\log 15}{\log 3}$ **Use the Change of Base Formula.**

≈ 2.4650 **Use a calculator.**

$\log_3 15 = \log_2 x$ **Write an equation.**

$2.4650 \approx \log_2 x$ **Substitute $\log_3 15 \approx 2.4650$.**

$x \approx 2^{2.4650}$ **Write in exponential form.**

≈ 5.5212 **Use a calculator.**

The expression $\log_3 15$ is approximately equal to 2.4650, or $\log_2 5.5212$.

Quick Check 5 **a.** Evaluate $\log_5 400$ and convert it to a logarithm in base 8. **3.7227, $\log_8 2301$**

b. Critical Thinking Consider the equation $2.465 \approx \log_2 x$ from Example 2. How could you solve the equation without using the Change of Base Formula?

Answers may vary. Sample: Use a calculator to raise 2 to the 2.465 power.

An equation that includes a logarithmic expression, such as $\log_3 15 = \log_2 x$ in Example 5, is called a **logarithmic equation.**

6 EXAMPLE Solving a Logarithmic Equation

Solve $\log(3x + 1) = 5$.

Method 1 $\log(3x + 1) = 5$

$3x + 1 = 10^5$ **Write in exponential form.**

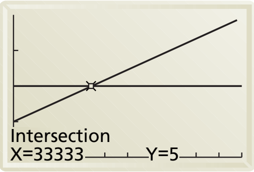
$3x + 1 = 100{,}000$

$x = 33{,}333$ **Solve for x.**

Method 2 Graph the equations $y_1 = \log(3x + 1)$ and $y_2 = 5$. Use Xmin = 30000, Xmax = 40000, Ymin = 4.9, Ymax = 5.1.

Find the point of intersection.

The solution is $x = 33{,}333$.



Method 3 Enter $y_1 = \log(3x + 1) - 5$.
Use tabular zoom-in to find the sign change. Use the information from Methods 1 or 2 to help you with your TblSet values.

The solution is $x = 33{,}333$.

X	Y1
33330	−4E−5
33331	−3E−5
33332	−1E−5
33333	0.0000
33334	1.3E−5
33335	2.6E−5
33336	3.9E−5

Y1=0

Check $\log(3x + 1) = 5$

$\log(3 \cdot 33{,}333 + 1) \stackrel{?}{=} 5$

$\log 100{,}000 \stackrel{?}{=} 5$

$\log 10^5 = 5$ ✓

Quick Check 6 Solve $\log(7 - 2x) = -1$. Check your answer. **3.45**

PowerPoint

Additional Examples

1 Solve $5^{2x} = 16$. **about 0.861**

2 Solve $4^{3x} = 1100$ by graphing. **$x \approx 1.684$**

3 Solve $5^{2x} = 120$ using tables. **about 1.49**

4 The population of trout in a certain stretch of the Platte River is shown for five consecutive years in the table, where 0 represents the year 1997. If the decay rate remains constant, in the beginning of which year might at most 100 trout remain in this stretch of river? **2015**

Time t	0	1	2	3	4
Pop. $P(t)$	5000	4000	3201	2561	2049

5 Use the Change of Base Formula to evaluate $\log_6 12$. Then convert $\log_6 12$ to a logarithm in base 3. **about 1.387; about $\log_3 4.589$**

Guided Instruction

7 EXAMPLE **Teaching Tip**

Ask a volunteer to explain why the logarithm of a negative number must be undefined.

PowerPoint

Additional Examples

6 Solve $\log(2x - 2) = 4$. **5001**

7 Solve $3 \log x - \log 2 = 5$. **about 58.48**

Resources

- Daily Notetaking Guide 8-5 L3
- Daily Notetaking Guide 8-5—Adapted Instruction L1

Closure

Ask students to give examples of equations that can be solved by using the properties of exponents and logarithms.

3. Practice

Assignment Guide

1 A B 1-24, 48-54, 58, 60-63, 66, 76-81, 85, 87, 89-91, 93, 95, 96

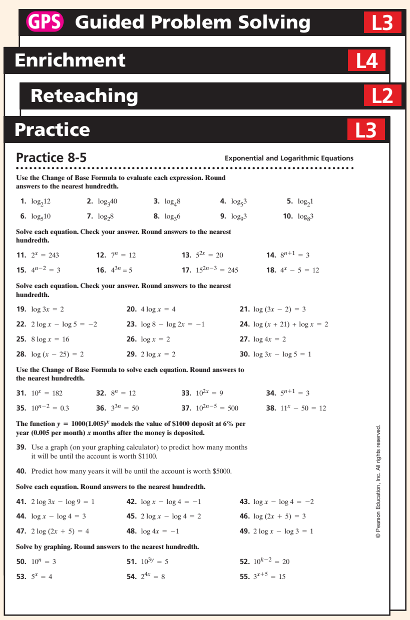
2 A B 25-47, 55-57, 59, 64, 65, 67-75, 82-84, 86, 88, 92, 94

C Challenge 97-105

Test Prep 106-112
Mixed Review 113-127

Homework Quick Check
To check students' understanding of key skills and concepts, go over Exercises 14, 47, 49, 64, 77, 87.

Differentiated Instruction Resources


GPS Guided Problem Solving L3
Enrichment L4
Reteaching L2
Practice L3

Practice 8-5 Exponential and Logarithmic Equations

Use the Change of Base Formula to evaluate each expression. Round answers to the nearest hundredth.

1. $\log_2 12$ 2. $\log_3 40$ 3. $\log_4 8$ 4. $\log_5 3$ 5. $\log_2 1$
6. $\log_5 10$ 7. $\log_2 8$ 8. $\log_3 6$ 9. $\log_9 3$ 10. $\log_8 3$

Solve each equation. Check your answer. Round answers to the nearest hundredth.

11. $2^x = 243$ 12. $7^n = 12$ 13. $5^{2x} = 20$ 14. $8^{n+1} = 3$
15. $4^{n-2} = 3$ 16. $4^{3n} = 5$ 17. $15^{2n-3} = 245$ 18. $4^x - 5 = 12$

Solve each equation. Check your answer. Round answers to the nearest hundredth.

19. $\log 3x = 2$ 20. $4 \log x = 4$ 21. $\log (3x - 2) = 3$
22. $2 \log x - \log 5 = -2$ 23. $\log 8 - \log 2x = -1$ 24. $\log (x + 21) + \log x = 2$
25. $8 \log x = 16$ 26. $\log x = 2$ 27. $\log 4x = 2$
28. $\log (x - 25) = 2$ 29. $2 \log x = 2$ 30. $\log 3x - \log 5 = 1$

Use the Change of Base Formula to solve each equation. Round answers to the nearest hundredth.

31. $10^x = 182$ 32. $8^n = 12$ 33. $10^{2x} = 9$ 34. $5^{n+1} = 3$
35. $10^{n-2} = 0.3$ 36. $3^{3n} = 50$ 37. $10^{2n-5} = 500$ 38. $11^x - 50 = 12$

The function $y = 1000(1.005)^x$ models the value of $1000 deposit at 6% per year (0.005 per month) x months after the money is deposited.

39. Use a graph (on your graphing calculator) to predict how many months it will be until the account is worth $1100.
40. Predict how many years it will be until the account is worth $5000.

Solve each equation. Round answers to the nearest hundredth.

41. $2 \log 3x - \log 9 = 1$ 42. $\log x - \log 4 = -1$ 43. $\log x - \log 4 = -2$
44. $\log x - \log 4 = 3$ 45. $2 \log x - \log 4 = 2$ 46. $\log (2x + 5) = 3$
47. $2 \log (2x + 5) = 4$ 48. $\log 4x = -1$ 49. $2 \log x - \log 3 = 1$

Solve by graphing. Round answers to the nearest hundredth.

50. $10^n = 3$ 51. $10^{3y} = 5$ 52. $10^{k-2} = 20$
53. $5^x = 4$ 54. $2^{4x} = 8$ 55. $3^{x+5} = 15$

Sometimes the properties of logarithms will help solve an equation.

7 EXAMPLE Using Logarithmic Properties to Solve an Equation

Solve $2 \log x - \log 3 = 2$.

$2 \log x - \log 3 = 2$

$\log\left(\frac{x^2}{3}\right) = 2$ **Write as a single logarithm.**

$\frac{x^2}{3} = 10^2$ **Write in exponential form.**

$x^2 = 3(100)$ **Multiply each side by 3.**

$x = \pm 10\sqrt{3}$, or about ± 17.32

Log x is defined only for $x > 0$, so the solution is $10\sqrt{3}$, or about 17.32.

Quick Check 7 Solve $\log 6 - \log 3x = -2$. **200**

EXERCISES

For more exercises, see *Extra Skill and Word Problem Practice.*

Practice and Problem Solving

A Practice by Example

GO for Help

Example 1 (page 461)

Solve each equation. Round to the nearest ten-thousandth. Check your answers.

1. $2^x = 3$ **1.5850** 2. $4^x = 19$ **2.1240** 3. $5^x = 81.2$ **2.7320** 4. $3^x = 27.3$ **3.0101**
5. $8 + 10^x = 1008$ **3** 6. $5 - 3^x = -40$ **3.4650** 7. $9^{2y} = 66$ **0.9534** 8. $4^{2z} = 40$ **1.330**
9. $14^{x+1} = 36$ **0.3579** 10. $12^{y-2} = 20$ **3.2056** 11. $25^{2x+1} = 144$ **0.2720** 12. $2^{3x-4} = 5$ **2.1073**

Example 2 (page 461)

Solve by graphing.

13. $4^{7x} = 250$ **0.5690** 14. $5^{3x} = 500$ **1.2871** 15. $6^x = 4565$ **4.7027** 16. $1.5^x = 356$ **14.4894**

Example 3 (page 462)

Use a table to solve each equation. Round to the nearest hundredth.

17. $2^{x+3} = 512$ **6** 18. $3^{x-1} = 72$ **4.89** 19. $6^{2x} = 10$ **0.64**
20. $3^{x-2} = 12x - 1$ **0.09, 5.86** 21. $4^{2x+1} = x^2$ **−0.73** 22. $2^{2x-1} = 3^x$ **2.41**

Example 4 (page 462)

23. An investment of $2000 earns 5.75% interest, which is compounded quarterly. After approximately how many years will the investment be worth $3000? **about 7.1 years**

24. The equation $y = 281(1.0124)^x$ models the U.S. population y, in millions of people, x years after the year 2000. Graph the function on your graphing calculator. Estimate when the U.S. population will reach 350 million. **about 2018**

Example 5 (page 463)

Use the Change of Base Formula to evaluate each expression. Then convert it to a logarithm in base 8. **25–32. See margin.**

25. $\log_2 9$ 26. $\log_4 8$ 27. $\log_3 54$ 28. $\log_5 62$
29. $\log_3 33$ 30. $\log_2 7$ 31. $\log_5 510$ 32. $\log_4 1.116$

Example 6 (page 463)

Solve each equation. Check your answers.

33. $\log 2x = -1$ **0.05** 34. $2 \log x = -1$ **34. $\frac{\sqrt{10}}{10}$ or about 0.3162** 35. $\log (3x + 1) = 2$ **33**
36. $\log x + 4 = 8$ **10,000** 37. $\log 6x - 3 = -4$ 38. $\log (x - 2) = 1$ **12**
39. $3 \log x = 1.5$ **$\sqrt{10}$, or about 3.1623** 40. $2 \log (x + 1) = 5$ 41. $\log (5 - 2x) = 0$ **2**

37. $\frac{1}{60}$, or ≈0.0167

40. $100\sqrt{10} - 1$, or ≈ 315.2

25. 3.1699; $\log_8 729$
26. 1.5; $\log_8 22.627$
27. 3.6309; $\log_8 1901.3$
28. 2.5643; $\log_8 206.93$
29. 3.1827; $\log_8 748.56$
30. 2.8074; $\log_8 343$
31. 3.8737; $\log_8 3149.6$
32. 0.0792; $\log_8 1.1790$

Example 7 (page 464)

Solve each equation.

42. $\log x - \log 3 = 8$ 3×10^8

43. $\log 2x + \log x = 11$ $100{,}000\sqrt{5}$, or about 223,606.8

44. $2 \log x + \log 4 = 2$ 5

45. $\log 5 - \log 2x = 1$ $\frac{1}{4}$

46. $3 \log x - \log 6 + \log 2.4 = 9$ 1357.2

47. $\log (7x + 1) = \log (x - 2) + 1$ 7

B Apply Your Skills

48. Consider the equation $2^{\frac{x}{3}} = 80$. a–c. See margin.

a. Solve the equation by taking the logarithm in base 10 of each side.

b. Solve the equation by taking the logarithm in base 2 of each side.

c. Writing Compare your result in parts (a) and (b). What are the advantages of either method? Explain.

Real-World Connection

Careers Seismologists use models to determine the source, nature, and size of seismic events.

49. **Seismology** An earthquake of magnitude 7.9 occurred in 2001 in Gujarat, India. It was 11,600 times as strong as the greatest earthquake ever to hit Pennsylvania. Find the magnitude of the Pennsylvania earthquake. (*Hint*: Refer to the Richter Scale on page 446.) 5.1

Write an equation. Then solve the equation without graphing.

50. A parent raises a child's allowance by 20% each year. If the allowance is \$8 now, when will it reach \$20? $20 = 8(1.2)^x$, 5 years

51. Protactinium-234*m*, a toxic radioactive metal with no known use, has a half-life of 1.17 minutes. How long does it take for a 10-mg sample to decay to 2 mg? See margin.

52. **Multiple Choice** As a town gets smaller, the population of its high school decreases by 12% each year. The student body has 125 students now. In how many years will it have about 75 students? A

Ⓐ 4 years Ⓑ 7 years Ⓒ 10 years Ⓓ 11 years

Mental Math **Solve each equation.**

53. $2^x = \frac{1}{2}$ -1

54. $3^x = 27$ 3

55. $\log_9 3 = x$ $\frac{1}{2}$

56. $\log_4 64 = x$ 3

57. $\log_8 2 = x$ $\frac{1}{3}$

58. $10^x = \frac{1}{100}$ -2

59. $\log_7 343 = x$ 3

60. $25^x = \frac{1}{5}$ $-\frac{1}{2}$

Population **Use this "Most Populous States" table for Exercises 61–63.**

Most Populous States

Rank in 2000	State	2000 Population	Average Annual Percentage Increase Since 1990
1	California	33,871,648	1.30%
2	Texas	20,851,820	2.08%
3	New York	18,976,457	0.54%
4	Florida	15,982,378	2.13%

SOURCE: U.S. Census Bureau. Go to www.PHSchool.com for a data update. Web Code: agg-9041

61. a. Determine the growth factors for Florida and New York. Then write an equation to model each state's population growth. a–b. See left.

b. Estimate when Florida's population might exceed New York's population.

62. a. Determine the growth factors for Texas and California. Then write an equation to model each state's population growth.

b. Estimate when Texas's population might exceed California's population. 2063

63. **Critical Thinking** Is it likely that Florida's population will exceed that of Texas? Explain your reasoning. See margin.

61a. Florida growth factor = 1.0213, $y = 15{,}982{,}378 \cdot (1.0213)^x$; New York growth factor = 1.0054, $y = 18{,}976{,}457 \cdot (1.0054)^x$

b. 2011

62a. Texas growth factor = 1.0208, $y = 20{,}851{,}820 \cdot (1.0208)^x$; California growth factor = 1.013, $y = 33{,}871{,}648 \cdot (1.013)^x$

Exercise 52 Show students that they can simply multiply the initial population of 125 by the multiplier 1 − 0.12 = 0.88, and determine the number of multiplications needed to get around 75. Alternatively, students can write an exponential equation that represents the situation and solve it.

Alternative Method

Exercise 53 Organize students in groups of 3. Have each student in the group try a different method: making a table to show successive values, graphing, or solving without graphing. Then have them discuss the relative merits of each method.

63. Since Florida's growth rate is larger than Texas's growth rate, in theory, given constant conditions, Florida would exceed Texas in about 543 years. However, since no state has unlimited capacity for growth, it is unrealistic to predict over a long period of time.

48a. 18.9658

b. 18.9658

c. Answers may vary. Sample: You don't have to use the change of base formula with the base-10 method, but there are fewer steps with the base-2 method.

51. $2 = 10\left(\frac{1}{2}\right)^{\frac{x}{1.17}}$, 2.7 min

Exercise 67 This exercise can make the Change of Base Formula seem less like an arbitrary rule, thereby making it easier to remember.

Auditory Learners

Exercise 97 Ask a volunteer to find a picture of piano strings (or actually take the class to see a piano if one is nearby) and discuss how the relative length of the strings relates to the pitch of each note on the keyboard. You could also use a guitar or a zither to demonstrate this.

65. Answers may vary. Sample: $\log x = 1.6$; $10^{1.6} = x, x \approx 39.81$

66. Answers may vary. Sample: A possible model is $y = 1465(1.0838)^x$ where $x = 0$ represents 1991; the growth is probably exponential and $1465(1.0838)^{10} \approx 3276$; using this model, there will be 10,000 manatees in about 2015.

67a. $x = \frac{\log b}{\log a}$

b. $x = \log_a b = \frac{\log b}{\log a}$

c. Substituting the result from part (a) into the results from part (b), or vice versa, yields $\log_a b = \frac{\log b}{\log a}$. This justifies the Change of Base Formula for $c = 10$.

77a. top up: 10^{-5} W/m^2, top down: $10^{-2.5}$ W/m^2

b. 99.68%

Real-World Connection

Many Florida manatees die after collisions with motorboats.

75. $\frac{\log (x + 1)}{\log x}$

GO Online Homework Help
Visit: PHSchool.com
Web Code: age-0805

64. Error Analysis What is wrong with the "proof" below that $2 = 1$? $\frac{\log 10^2}{\log 10^1} \neq \log 10^{2-1}$

$$2 = \frac{2}{1} = \frac{\log 10^2}{\log 10^1} = \log 10^{2-1} = \log 10^1 = 1$$

65. Open-Ended Write and solve a logarithmic equation. **See margin.**

66. Zoology Conservation efforts have increased the endangered Florida manatee population from 1465 in 1991 to 3276 in 2001. If this growth rate continues, when might there be 10,000 manatees? Explain the reasoning behind your choice of a model. **See margin.**

67. Consider the equation $a^x = b$.
a. Solve the equation by using log base 10. **a–c. See margin.**
b. Solve the equation by using log base a.
c. Use your results in parts (a) and (b) to justify the Change of Base Formula.

Write each logarithm as the quotient of two common logarithms. Do not simplify the quotient.

68. $\log_7 2$ $\frac{\log 2}{\log 7}$ **69.** $\log_3 8$ $\frac{\log 8}{\log 3}$ **70.** $\log_5 140$ $\frac{\log 140}{\log 5}$ **71.** $\log_9 3.3$ $\frac{\log 3.3}{\log 9}$

72. $\log_4 3x$ $\frac{\log 3x}{\log 4}$ **73.** $\log_6 (1 - x)$ $\frac{\log (1 - x)}{\log 6}$ **74.** $\log_x 5$ $\frac{\log 5}{\log x}$ **75.** $\log_x (x + 1)$ **See left.**

Acoustics In Exercises 76–78, the loudness measured in decibels (dB) is defined by loudness = $10 \log \frac{I}{I_0}$, where I is the intensity and $I_0 = 10^{-12}$ W/m^2.

76. The human threshold for pain is 120 dB. Instant perforation of the eardrum occurs at 160 dB.
a. Find the intensity of each sound. 10^0 (or 1) W/m^2, 10^4 W/m^2
b. How many times as intense is the noise that will perforate an eardrum as the noise that causes pain? 10,000 times as intense

77. The noise level inside a convertible driving along the freeway with its top up is 70 dB. With the top down, the noise level is 95 dB. **a–b. See margin.**
a. Find the intensity of the sound with the top up and with the top down.
b. By what percent does leaving the top up reduce the intensity of the sound?

78. A screaming child can reach 90 dB. A launch of the space shuttle produces sound of 180 dB at the launch pad.
a. Find the intensity of each sound. 10^{-3} W/m^2, 10^6 W/m^2
b. How many times as intense as the noise from a screaming child is the noise from a shuttle launch? 10^9 times more intense

Solve each equation. If necessary, round to the nearest ten-thousandth.

79. $8^x = 444$ 2.9315 **80.** $14^{9x} = 146$ 0.2098

81. $3^{7x} = 120$ 0.6225 **82.** $\frac{1}{2}\log x + \log 4 = 2$ 625

83. $4 \log_3 2 - 2 \log_3 x = 1$ 2.3094 **84.** $\log x^2 = 2$ 10

85. $9^{2x} = 42$ 0.8505 **86.** $\log_8 (2x - 1) = \frac{1}{3}$ 1.5

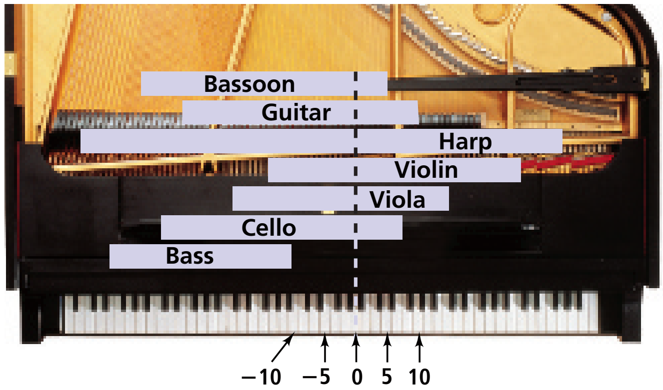
87. $1.3^x = 7$ 7.4168 **88.** $\log (5x - 4) = 3$ 200.8

89. $2.1^x = 9$ 2.9615 **90.** $12^{4 - x} = 20$ 2.7944

91. $5^{3x} = 125$ 1 **92.** $\log 4 + 2 \log x = 6$ 500

93. $4^{3x} = 77.2$ 1.0451 **94.** $\log_7 3x = 3$ $114.\overline{3}$

95. $3^x + 0.7 = 4.9$ 1.3063 **96.** $7^x - 1 = 371$ 3.0417



Challenge

97a. bassoon, guitar, harp, violin, viola, cello
b. bassoon, guitar, harp, cello, bass
c. harp, violin
d. harp, violin

97. Music The pitch, or frequency, of a piano note is related to its position on the keyboard by the function $F(n) = 440 \cdot 2^{\frac{n}{12}}$, where F is the frequency of the sound wave in cycles per second and n is the number of piano keys above or below Concert A, as shown above. If $n = 0$ at Concert A, which of the instruments shown in the diagram can sound notes of the given frequency?

a. 590 **b.** 120 **c.** 1440 **d.** 2093

98. Astronomy The brightness of an astronomical object is called its magnitude. A decrease of five magnitudes increases the brightness exactly 100 times. The sun is magnitude −26.7, and the full moon is magnitude −12.5. The sun is about how many times brighter than the moon? **478,630 times**

99. Archaeology A scientist carbon-dates a piece of fossilized tree trunk that is thought to be over 5000 years old. The scientist determines that the sample contains 65% of the original amount of carbon-14. The half-life of carbon-14 is 5730 years. Is the reputed age of the tree correct? Explain. **No; solving $0.65 = (0.5)^{\frac{x}{5730}}$ for x, the age in years of the sample, yields an age of about 3561 yr.**

Solve each equation.

100. $\log_7 (2x - 3)^2 = 2$ **5**

101. $\log_2 (x^2 + 2x) = 3$ **−4, 2**

102. $\log_4 (x^2 - 17) = 3$ **−9, 9**

103. $\frac{3}{2} \log_2 4 - \frac{1}{2} \log_2 x = 3$ **1**

104. 20,031 m above sea level
105b. 0.928 mg or 1.061 mg
c. Estimate in hours is more accurate; the days have a larger rounding error.

104. In the formula $P = P_0\left(\frac{1}{2}\right)^{\frac{h}{4795}}$, P is the atmospheric pressure in millimeters of mercury at elevation h meters above sea level. P_0 is the atmospheric pressure at sea level. If P_0 equals 760 mm, at what elevation is the pressure 42 mm?

105. Chemistry A technician found 12 mg of a radon isotope in a soil sample. After 24 hours, another measurement revealed 10 mg of the isotope.

a. Estimate the length of the isotope's half-life to the nearest hour and to the nearest day. **91 hours or 4 days**

b. For each estimate, determine the amount of the isotope after two weeks.

c. Compare your answers to part (b). Which is more accurate? Explain.

Test Prep

Gridded Response

Use a calculator to solve each equation. Enter each answer to the nearest hundredth.

106. $7^{2x} = 75$ **1.11**

107. $11^{x-5} = 250$ **7.30**

108. $1080 = 15^{3x-4}$ **2.19**

4. Assess & Reteach

PowerPoint

Lesson Quiz

Use mental math to solve each equation.

1. $2^x = \frac{1}{8}$ **−3**
2. $\log_4 2 = x$ $\frac{1}{2}$
3. $10^{6x} = 1$ **0**
4. Solve $5^{2x} = 125$. $\frac{3}{2}$

Alternative Assessment

Have a class discussion. Ask students to talk about which aspect of solving exponential and logarithmic equations they found most confusing or difficult, citing specific examples from the lesson. Have other students give tips and explanations to help clarify the topics. Be sure each student contributes to the discussion.

Test Prep

A sheet of blank grids is available in the Test-Taking Strategies with Transparencies booklet. Give this sheet to students for practice with filling in the grids.

Resources

For additional practice with a variety of test item formats:

- Standardized Test Prep, p. 483
- Test-Taking Strategies, p. 478
- Test-Taking Strategies with Transparencies

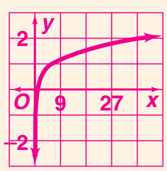
Checkpoint Quiz

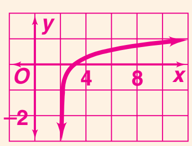
Use this Checkpoint Quiz to check students' understanding of the skills and concepts of Lessons 8-3 through 8-5.

Resources
Grab & Go
- Checkpoint Quiz 2

113. $\log 2 + 3 \log x - 2 \log y$
114. $\log_3 x - \log_3 y$
115. $3 \log_2 3 + 3 \log_2 x$
116. $\log_3 7 + 2 \log_3 (2x - 3)$
117. $\log_4 5 + \frac{1}{2} \log_4 x$
118. $\log_2 5 + \log_2 a - 2 \log_2 b$

Checkpoint Quiz

1.


2.


4. $2 \log_6 3 + 2 \log_6 x + 2 \log_6 y$
5. $\log_6 4 + \frac{1}{2} \log_6 x$
10. Rewrite $\log_2 10$ as $\frac{\log 10}{\log 2}$ and evaluate it to get ≈ 3.322. Then set $3.322 = \log_3 x$. Rewrite to get $3.322 = \frac{\log x}{\log 3}$ and solve. Convert $\log x = 1.585$ to $10^{1.585} = x$ or $x \approx 38.46$. So $\log_2 10 \approx \log_3 38.46$.

Gridded Response Use the Change of Base Formula to solve each equation. Enter the answer to the nearest tenth.

109. $\log_5 x = \log_3 20$ **80.5**　　**110.** $\log_9 x = \log_6 15$ **27.7**

Solve each equation.

111. $\log (1 + 3x) = 3$ **333**　　**112.** $\log (x - 3) = 2$ **103**

Mixed Review

GO for Help

Lesson 8-4 **Expand each logarithm.** **113–118. See margin.**

113. $\log 2x^3y^{-2}$　　**114.** $\log_3 \frac{x}{y}$　　**115.** $\log_2 (3x)^3$

116. $\log_3 7(2x - 3)^2$　　**117.** $\log_4 5\sqrt{x}$　　**118.** $\log_2 \left(\frac{5a}{b^2}\right)$

Lesson 7-6 **Let $f(x) = 3x$ and $g(x) = x^2 - 1$. Perform each function operation.**

119. $(f + g)(x)$ $x^2 + 3x - 1$　　**120.** $(g - f)(x)$ $x^2 - 3x - 1$　　**121.** $(f \cdot g)(x)$ $3x^3 - 3x$

Lesson 6-6 **Find all the zeros of each function.**

122. $y = x^3 - x^2 + x - 1$ **1, $\pm i$**　　**123.** $f(x) = x^4 - 16$ **$\pm 2, \pm 2i$**

124. $f(x) = x^4 - 5x^2 + 6$ **$\pm\sqrt{2}, \pm\sqrt{3}$**　　**125.** $y = 3x^3 - 21x - 18$ **$-2, -1, 3$**

Lesson 1-3 **Write an equation to solve each problem.**

126. A customer at a hardware store mentions that he is buying fencing for a vegetable garden that is 12 ft longer than it is wide. He buys 128 ft of fencing. What is the width of the garden? **$2(x) + 2(x + 12) = 128$; 26 ft**

127. A bowler has an average of 133. In a set of games one night, her scores are 135, 127, 119, 142, and 156. What score must she bowl in the sixth game to maintain her average? **$\frac{679 + x}{6} = 133$; 119**

Checkpoint Quiz 2 — Lessons 8-3 through 8-5

Graph each logarithmic function. **1–2. See margin.**

1. $y = \log_6 x$　　**2.** $y = \log (x - 2)$

Expand each logarithm. **4–5. See margin.**

3. $\log \frac{s^3}{r^5}$ **$3 \log s - 5 \log r$**　　**4.** $\log_6 (3xy)^2$　　**5.** $\log_6 4\sqrt{x}$

Solve each equation.

6. $7 - 2^x = -1$ **3**　　**7.** $\log 5x = 2$ **20**　　**8.** $3 \log x = 9$ **1000**

9. Evaluate the expressions below and order them from least to greatest.
2^3　　$\log_2 3$　　$\log_3 2$　　3^2　　$\log 2$

9. $\log 2 = 0.3010$, $\log_3 2 = 0.6309$, $\log_2 3 = 1.5850$, $2^3 = 8$, $3^2 = 9$

10. Writing Explain how to use the Change of Base Formula to rewrite $\log_2 10$ as a logarithmic expression with base 3. **See margin.**

8-6

1. Plan

Objectives
1 To evaluate natural logarithmic expressions
2 To solve equations using natural logarithms

Examples
1 Simplifying Natural Logarithms
2 Real-World Connection
3 Solving a Natural Logarithmic Equation
4 Solving an Exponential Equation
5 Real-World Connection

Professional Development

Math Background

Natural logarithms, or logarithms with base e, occur in many problems involving the growth and decay of natural organisms. In fact, natural logarithms arise in those problems which model continuous growth and decay as discussed in Lesson 8-2.

More Math Background: p. 428D

Lesson Planning and Resources

See p. 428E for a list of the resources that support this lesson.

PowerPoint

Bell Ringer Practice

Check Skills You'll Need
For intervention, direct students to:

Properties of Exponential Functions
Lesson 8-2: Example 4
Extra Skills and Word Problems Practice, Ch. 8

Exponential and Logarithmic Equations
Lesson 8-5: Example 6
Extra Skills and Word Problems Practice, Ch. 8

8-6

Natural Logarithms

What You'll Learn

- To evaluate natural logarithmic expressions
- To solve equations using natural logarithms

. . . And Why

To model the velocity of a rocket, as in Example 2

Check Skills You'll Need GO for Help Lessons 8-2 and 8-5

Use your calculator to evaluate each expression to the nearest thousandth.

1. e^5 148.413 **2.** $2e^3$ 40.171 **3.** e^{-2} 0.135 **4.** $\frac{1}{e}$ 0.368 **5.** $4.2e$ 11.417

Solve.

6. $\log_3 x = 4$ 81 **7.** $\log_{16} 4 = x$ $\frac{1}{2}$ **8.** $\log_{16} x = 4$ 65,536

New Vocabulary • natural logarithmic function

1 Natural Logarithms

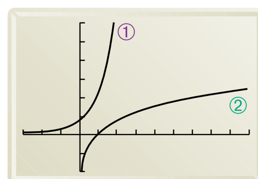
In Lesson 8-2, you learned that the number $e \approx 2.71828$ can be used as a base for exponents. The function $y = e^x$ has an inverse, the **natural logarithmic function.**

Key Concepts

Definition **Natural Logarithmic Function**

If $y = e^x$, then $\log_e y = x$, which is commonly written as $\ln y = x$.

The natural logarithmic function is the inverse, written as $y = \ln x$.



① $y = e^x$
② $y = \ln x$

Vocabulary Tip

ln y means "the natural logarithm of y." The l stands for "logarithm" and the n stands for "natural."

The properties of common logarithms apply to natural logarithms also.

1 EXAMPLE Simplifying Natural Logarithms

Write $3 \ln 6 - \ln 8$ as a single natural logarithm.

$3 \ln 6 - \ln 8 = \ln 6^3 - \ln 8$ **Power Property**

$= \ln \frac{6^3}{8}$ **Quotient Property**

$= \ln 27$ **Simplify.**

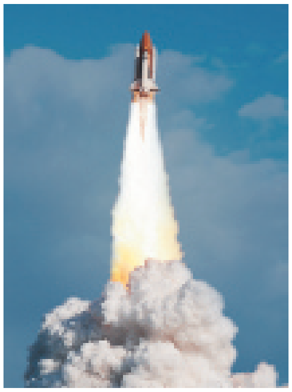
Quick Check 1 Write each expression as a single natural logarithm.

a. $5 \ln 2 - \ln 4$ ln 8 **b.** $3 \ln x + \ln y$ $\ln x^3y$ **c.** $\frac{1}{4} \ln 3 + \frac{1}{4} \ln x$ $\ln \sqrt[4]{3x}$

Differentiated Instruction **Solutions for All Learners**

Special Needs L1
In Example 4, students may fail to take the natural logarithm of both sides. Stress that if any logarithm is taken on one side of the equation, the same logarithm must be applied to the other side. Ask: *Why was the natural logarithm used?* base is e
learning style: verbal

Below Level L2
Review the general formula for growth or decay, $N = N_0e^{kt}$ and the formula for continuously compounded interest, $A = Pe^{rt}$. Compare the meanings of corresponding variables.
learning style: verbal



Real-World Connection

The space shuttle is launched into orbit.

Natural logarithms are useful because they help express many relationships in the physical world.

2 EXAMPLE Real-World Connection

Space A spacecraft can attain a stable orbit 300 km above Earth if it reaches a velocity of 7.7 km/s. The formula for a rocket's maximum velocity v in kilometers per second is $v = -0.0098t + c \ln R$. The booster rocket fires for t seconds and the velocity of the exhaust is c km/s. The ratio of the mass of the rocket filled with fuel to its mass without fuel is R. Suppose a rocket used to propel a spacecraft has a mass ratio of 25, an exhaust velocity of 2.8 km/s, and a firing time of 100 s. Can the spacecraft attain a stable orbit 300 km above Earth?

Let $R = 25$, $c = 2.8$, and $t = 100$. Find v.

$v = -0.0098t + c \ln R$ **Use the formula.**

$= -0.0098(100) + 2.8 \ln 25$ **Substitute.**

$\approx -0.98 + 2.8(3.219)$ **Use a calculator.**

≈ 8.0 **Simplify.**

The maximum velocity of 8.0 km/s is greater than the 7.7 km/s needed for a stable orbit. Therefore, the spacecraft can attain a stable orbit 300 km above Earth.

Quick Check 2 **a.** A booster rocket for a spacecraft has a mass ratio of about 15, an exhaust velocity of 2.1 km/s, and a firing time of 30 s. Find the maximum velocity of the spacecraft. Can the spacecraft achieve a stable orbit 300 km above Earth? ≈5.4 km/s; no

b. Critical Thinking Suppose a rocket, as designed, cannot provide enough velocity to achieve a stable orbit. Look at the variables in the velocity formula. What alterations could be made to the rocket so that a stable orbit could be achieved? **One could increase its mass ratio or its exhaust velocity.**

2 Natural Logarithmic and Exponential Equations

You can use the properties of logarithms to solve natural logarithmic equations.

3 EXAMPLE Solving a Natural Logarithmic Equation

Solve $\ln (3x + 5)^2 = 4$.

$\ln (3x + 5)^2 = 4$

$(3x + 5)^2 = e^4$ **Rewrite in exponential form.**

$(3x + 5)^2 \approx 54.60$ **Use a calculator.**

$3x + 5 \approx \pm\sqrt{54.60}$ **Take the square root of each side.**

$3x + 5 \approx 7.39$ or -7.39 **Use a calculator.**

$x \approx 0.797$ or -4.130 **Solve for x.**

Check $\ln (3 \cdot 0.797 + 5)^2 \stackrel{?}{=} 4$ $\quad$ $\ln (3 \cdot (-4.130) + 5)^2 \stackrel{?}{=} 4$

$\ln 54.6 \stackrel{?}{=} 4$ $\quad$ $\ln 54.6 \stackrel{?}{=} 4$

$4.000 \approx 4$ ✓ $\quad$ $4.000 \approx 4$ ✓

Quick Check 3 Solve each equation. Check your answers.

a. $\ln x = 0.1$ 1.105 **b.** $\ln (3x - 9) = 21$ 439,605,247.8 **c.** $\ln\left(\frac{x+2}{3}\right) = 12$ 488,262.4

2. Teach

Guided Instruction

2 EXAMPLE Teaching Tip

Point out that the speed is in kilometers per second. To put this in more familiar units so students can get a sense of how fast the rocket is moving, suggest that they convert this speed to km/h or mi/h.

PowerPoint

Additional Examples

1 Write $2 \ln 12 - \ln 9$ as a single natural logarithm. **ln 16**

2 Find the velocity of a spacecraft whose booster rocket has a mass ratio of 22, an exhaust velocity of 2.3 km/s, and a firing time of 50 s. Can the spacecraft achieve a stable orbit 300 km above Earth? **about 6.6 km/s; no**

Guided Instruction

3 EXAMPLE Error Prevention

Remind students that the parentheses in $\ln (3x + 5)$ are necessary in order to show that you are taking the natural logarithm of the entire binomial.

Advanced Learners L4

Ask students to solve $3 \cdot 7^{2x} + 3 = 14$ by taking logs to the base 7 and using the change of base formula.

learning style: verbal

English Language Learners ELL

Help students make the connection between natural logarithms and the math term **ln**. Discuss the pronunciation of ln as the letters sound, or "ell n." Write $\ln x = \log_e x$ on the board and have students read it to emphasize the meaning of natural logarithms.

learning style: verbal

Additional Examples

3 Solve $\ln(2x - 4)^3 = 6$. **about 5.695**

4 Use natural logarithms to solve $4e^{3x} + 1.2 = 14$. **about 0.388**

5 An initial investment of $200 is now valued at $254.25. The interest rate is 6%, compounded continuously. How long has the money been invested? **about 4 years**

4 EXAMPLE Technology Tip

Have students check to see if their calculator has separate keys for LOG and LN.

Resources

- Daily Notetaking Guide 8-6 L3
- Daily Notetaking Guide 8-6—Adapted Instruction L1

Closure

Ask students to compare and contrast natural and common logarithms. **Answers may vary. Sample: They are both exponents; they both obey the same properties of exponents and logarithms; they use a different base. The base for common logarithms is the rational number 10; for natural logarithms, it is the irrational number *e*.**

You can use natural logarithms to solve exponential equations.

4 EXAMPLE Solving an Exponential Equation

Use natural logarithms to solve $7e^{2x} + 2.5 = 20$.

$7e^{2x} + 2.5 = 20$

$7e^{2x} = 17.5$ — **Subtract 2.5 from each side.**

$e^{2x} = 2.5$ — **Divide each side by 7.**

$\ln e^{2x} = \ln 2.5$ — **Take the natural logarithm of each side.**

$2x = \ln 2.5$ — **Simplify.**

$x = \frac{\ln 2.5}{2}$ — **Solve for *x*.**

$x \approx 0.458$ — **Use a calculator.**

Quick Check **4** Use natural logarithms to solve each equation.

a. $e^{x+1} = 30$ **2.401** **b.** $e^{\frac{2x}{5}} + 7.2 = 9.1$ **1.605**

Test-Taking Tip

If you perform an operation on one side of an equation, remember to perform the same operation on the other side.

5 EXAMPLE Real-World Connection

Multiple Choice An investment of $100 is now valued at $149.18. The interest rate is 8%, compounded continuously. About how long has the money been invested?

(A) 2 years (B) 5 years (C) 7 years (D) 19 years

$A = Pe^{rt}$ — **Continuously compounded interest formula**

$149.18 = 100e^{0.08t}$ — **Substitute 149.18 for *A*, 100 for *P*, and 0.08 for *r*.**

$1.4918 = e^{0.08t}$ — **Divide each side by 100.**

$\ln 1.4918 = \ln e^{0.08t}$ — **Take the natural logarithm of each side.**

$\ln 1.4918 = 0.08t$ — **Simplify.**

$\frac{\ln 1.4918}{0.08} = t$ — **Solve for *t*.**

$5 \approx t$ — **Use a calculator.**

The money has been invested for about five years. The answer is B.

Quick Check **5** An initial investment of $200 is worth $315.24 after seven years of continuous compounding. Find the interest rate. **6.5%**

EXERCISES

For more exercises, see *Extra Skill and Word Problem Practice.*

Practice and Problem Solving

A Practice by Example

GO for Help — Example 1 (page 470)

Write each expression as a single natural logarithm.

1. $3 \ln 5$ **ln 125**
2. $\ln 9 + \ln 2$ **ln 18**
3. $\ln 24 - \ln 6$ **ln 4**
4. $4 \ln 8 + \ln 10$ **ln 40,960**
5. $\ln 3 - 5 \ln 3$ **$\ln \frac{1}{81}$**
6. $2 \ln 8 - 3 \ln 4$ **ln 1**
7. $5 \ln m - 3 \ln n$ **$\ln \frac{m^5}{n^3}$**
8. $\frac{1}{3}(\ln x + \ln y) - 4 \ln z$ **$\ln \frac{\sqrt[3]{xy}}{z^4}$**
9. $\ln a - 2 \ln b + \frac{1}{2} \ln c$ **$\ln \frac{a\sqrt{c}}{b^2}$**

Example 2 (page 471)

Find the value of *y* for the given value of *x*.

10. $y = 15 + 3 \ln x$, for $x = 7.2$ **20.92**
11. $y = 0.05 - 10 \ln x$, for $x = 0.09$ **24.13**

For Exercises 12 and 13, use $v = -0.0098t + c \ln R$.

12. **Space** Find the velocity of a spacecraft whose booster rocket has a mass ratio of 20, an exhaust velocity of 2.7 km/s, and a firing time of 30 s. Can the spacecraft achieve a stable orbit 300 km above Earth? **7.79 km/s; yes**

13. A rocket has a mass ratio of 24 and an exhaust velocity of 2.5 km/s. Determine the minimum firing time for a stable orbit 300 km above Earth. **25 s**

Example 3 (page 471)

Solve each equation. Check your answers.

14. $\ln 3x = 6$ **134.476**
15. $\ln x = -2$ **0.135**
16. $\ln (4x - 1) = 36$ **1.078×10^{15}**
17. $\ln (2m + 3) = 8$ **1488.979**
18. $\ln (t - 1)^2 = 3$ **5.482, −3.482**
19. $1.1 + \ln x^2 = 6$ **±11.588**
20. $\ln \frac{x - 1}{2} = 4$ **110.196**
21. $\ln 4r^2 = 3$ **±2.241**
22. $2 \ln 2x^2 = 1$ **±0.908**

Example 4 (page 472)

Use natural logarithms to solve each equation.

23. $e^x = 18$ **2.890**
24. $e^{2x} = 10$ **1.151**
25. $e^{x+1} = 30$ **2.401**
26. $e^{\frac{x}{5}} + 4 = 7$ **5.493**
27. $e^{2x} = 12$ **1.242**
28. $e^{\frac{x}{9}} - 8 = 6$ **23.752**

Example 5 (page 472)

29. **Investing** An initial deposit of \$200 is now worth \$331.07. The account earns 8.4% interest, compounded continuously. Determine how long the money has been in the account. **6 years**

30. An investor sold 100 shares of stock valued at \$34.50 per share. The stock was purchased at \$7.25 per share two years ago. Find the rate of continuously compounded interest that would be necessary in a banking account for the investor to make the same profit. **78%**

B Apply Your Skills

Mental Math **Simplify each expression.**

31. $\ln e$ **1**
32. $\ln e^2$ **2**
33. $\ln e^{10}$ **10**
34. $10 \ln e$ **10**
35. $\ln 1$ **0**
36. $\frac{\ln e}{4}$ **$\frac{1}{4}$**
37. $\frac{\ln e^2}{2}$ **1**
38. $\ln e^{83}$ **83**

GO for Help

For Gridded Responses, see Test-Taking Strategies, page 46.

39. **Gridded Response** The battery power available to run a satellite is given by the formula $P = 50e^{-\frac{t}{250}}$, where P is power in watts and t is time in days. For how many days can the satellite run if it requires 15 watts? **301**

40. **Space** Use the formula for maximum velocity $v = -0.0098t + c \ln R$. Find the mass ratio of a rocket with an exhaust velocity of 3.1 km/s, a firing time of 50 s, and a maximum shuttle velocity of 6.9 km/s. **10.8**
GPS

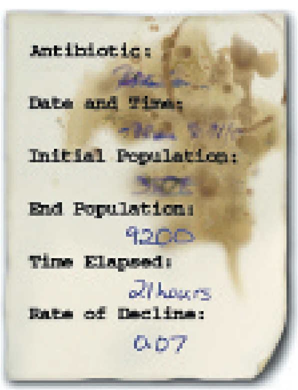
Determine whether each statement is *always* true, *sometimes* true, or *never* true.

41. $\ln e^x > 1$ **sometimes**
42. $\ln e^x = \ln e^x + 1$ **never**
43. $\ln t = \log_e t$ **always**

Biology **For Exercises 44–46, use the formula $H = \left(\frac{1}{r}\right)(\ln P - \ln A)$. H is the number of hours, r is the rate of decline, P is the initial bacteria population, and A is the reduced bacteria population.**

44. A scientist determines that an antibiotic reduces a population of 20,000 bacteria to 5000 in 24 hours. Find the rate of decline caused by the antibiotic. **about 5.8% per hour**

45. A laboratory assistant tests an antibiotic that causes a rate of decline of 0.14. How long should it take for a population of 8000 bacteria to shrink to 500? **about 19.8 h**



46. A scientist spilled coffee on the lab report shown at the left. Determine the initial population of the bacteria. **about 40,000 bacteria**

3. Practice

Assignment Guide

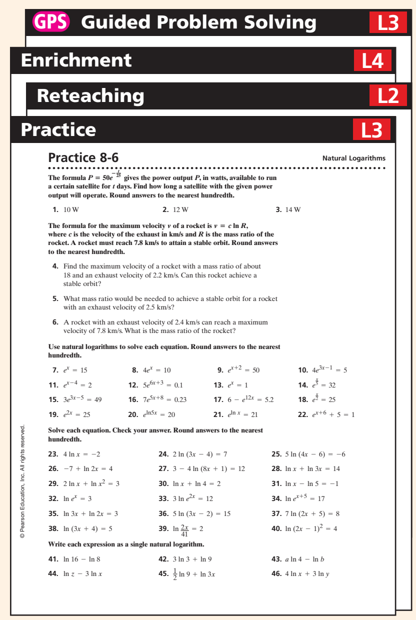
1 A B	1-13, 31-38	
2 A B	14-30, 39-62	
C Challenge		63-66
Test Prep		67-70
Mixed Review		71-80

Homework Quick Check
To check students' understanding of key skills and concepts, go over Exercises 8, 18, 30, 39, 44, 56.

Alternative Method

Exercises 31–38 Suggest that students ask themselves the question in a different form. For example, for Exercise 32 ask: *What power of the base e gives me the number e^2?*

Differentiated Instruction Resources


GPS Guided Problem Solving L3
Enrichment L4
Reteaching L2
Practice L3

Practice 8-6 Natural Logarithms

The formula $P = 50e^{-\frac{t}{25}}$ gives the power output P, in watts, available to run a certain satellite for t days. Find how long a satellite with the given power output will operate. Round answers to the nearest hundredth.

1. 10 W
2. 12 W
3. 14 W

The formula for the maximum velocity v of a rocket is $v = c \ln R$, where c is the velocity of the exhaust in km/s and R is the mass ratio of the rocket. A rocket must reach 7.8 km/s to attain a stable orbit. Round answers to the nearest hundredth.

4. Find the maximum velocity of a rocket with a mass ratio of about 18 and an exhaust velocity of 2.2 km/s. Can this rocket achieve a stable orbit?
5. What mass ratio would be needed to achieve a stable orbit for a rocket with an exhaust velocity of 2.5 km/s?
6. A rocket with an exhaust velocity of 2.4 km/s can reach a maximum velocity of 7.8 km/s. What is the mass ratio of the rocket?

Use natural logarithms to solve each equation. Round answers to the nearest hundredth.

7. $e^x = 15$
8. $4e^x = 10$
9. $e^{x+2} = 50$
10. $4e^{3x-1} = 5$
11. $e^{x-4} = 2$
12. $5e^{6x+3} = 0.1$
13. $e^x = 1$
14. $e^{\frac{x}{5}} = 32$
15. $3e^{3x-5} = 49$
16. $7e^{5x+8} = 0.23$
17. $6 - e^{12x} = 5.2$
18. $e^{\frac{x}{2}} = 25$
19. $e^{2x} = 25$
20. $e^{\ln 5x} = 20$
21. $e^{\ln x} = 21$
22. $e^{x+6} + 5 = 1$

Solve each equation. Check your answer. Round answers to the nearest hundredth.

23. $4 \ln x = -2$
24. $2 \ln (3x - 4) = 7$
25. $5 \ln (4x - 6) = -6$
26. $-7 + \ln 2x = 4$
27. $3 - 4 \ln (8x + 1) = 12$
28. $\ln x + \ln 3x = 14$
29. $2 \ln x + \ln x^2 = 3$
30. $\ln x + \ln 4 = 2$
31. $\ln x - \ln 5 = -1$
32. $\ln e^x = 3$
33. $3 \ln e^{2x} = 12$
34. $\ln e^{x+5} = 17$
35. $\ln 3x + \ln 2x = 3$
36. $5 \ln (3x - 2) = 15$
37. $7 \ln (2x + 5) = 8$
38. $\ln (3x + 4) = 5$
39. $\ln \frac{2x}{41} = 2$
40. $\ln (2x - 1)^2 = 4$

Write each expression as a single natural logarithm.

41. $\ln 16 - \ln 8$
42. $3 \ln 3 + \ln 9$
43. $a \ln 4 - \ln b$
44. $\ln z - 3 \ln x$
45. $\frac{1}{2} \ln 9 + \ln 3x$
46. $4 \ln x + 3 \ln y$

4. Assess & Reteach

PowerPoint

Lesson Quiz

1. Write 4 ln 6 − 2 ln 3 as a single natural logarithm. **ln 144**
2. Solve $e^{3x} = 15$. **about 0.903**
3. Simplify $\ln e^7$. **7**

Alternative Assessment

Ask students to discuss with a partner the differences in the procedures and answers for solving these two equations: $\ln x - \ln(x-1) = 2$ and $\log x - \log(x-1) = 2$. Then have each pair write a paragraph that summarizes their discussion. **The procedures are the same. If b is the base for the logarithm, then the answer for both equations is $b^2 \div (b^2 - 1)$. However, for the first equation, $b = e$ and for the second equation, $b = 10$. The answers are about 1.157 and about 1.010.**

63. No; using the Change of Base Formula would result in one of the log expressions being written as a quotient of logs, which couldn't then be combined with the other expression to form a single logarithm.

64d. $t = \dfrac{\ln\left(\frac{y}{300}\right)}{0.241}$**, where y is the number of Internet users in millions and t is time in years.**

e. Substitute the number of users found in (b) and (c) into the equation in (d). Determine whether your answers in years are the same as t for each.

Savings Suppose you invest \$500 at 5% interest compounded continuously. Copy and complete the table to find how long it will take to reach each amount.

	Amount (A)	Time (years)	
47.	\$600	■	**47. 3.6**
48.	\$700	■	**48. 6.7**
49.	\$800	■	**49. 9.4**
50.	\$900	■	**50. 11.8**
51.	\$1000	■	**51. 13.9**
52.	\$1100	■	**52. 15.8**
53.	\$1200	■	**53. 17.5**
54.	\$1300	■	**54. 19.1**

GO Online Homework Help
Visit: PHSchool.com
Web Code: age-0808

Solve each equation.

55. $\ln x - 3\ln 3 = 3$ **542.31**
56. $\ln(2x - 1) = 0$ **1**
57. $4e^{x+2} = 32$ **0.0794**
58. $\ln(5x - 3)^{\frac{1}{3}} = 2$
59. $2e^{3x-2} + 4 = 16$
60. $2e^{x-2} = e^x + 7$
61. $\frac{1}{3}\ln x + \ln 2 - \ln 3 = 3$ **27,347.9**
62. $\ln(x + 2) - \ln 4 = 3$ **78.342**

58. 81.286
59. 1.2639
60. no solution

C Challenge

63. Critical Thinking Can $\ln 5 + \log_2 10$ be written as a single logarithm? Explain. **See margin.**

64. In 2000, there were about 300 million Internet users. That number is projected to grow to 1 billion in 2005. **$y = 300e^{0.241t}$**

a. Let t represent the time, in years, since 2000. Write a function of the form $y = ae^{ct}$ that models the expected growth in the population of Internet users.
b. In what year might there be 500 million Internet users? **2002**
c. In what year might there be 1.5 billion Internet users? **2006**
d. Solve your equation for t. **d–e. See margin.**
e. Writing Explain how you can use your equation from part (d) to verify your answers to parts (b) and (c).

65. Physics The function $T(t) = T_r + (T_i - T_r)e^{kt}$ models Newton's Law of Cooling. $T(t)$ is the temperature of a heated substance t minutes after it has been removed from a heat (or cooling) source. T_i is the substance's initial temperature, k is a constant for that substance, and T_r is room temperature.

a. The initial surface temperature of a beef roast is 236°F and room temperature is 72°F. If $k = -0.041$, how long will it take for this roast to cool to 100°F?
b. Write and graph an equation that you can use to check your answer to part (a). Use your graph to complete the table below. **a–b. See margin.**

Temperature (°F)	225	200	175	150	125	100	75
Minutes Later	■	■	■	■	■	■	■
	1.7	**6.0**	**11.3**	**18.1**	**27.6**	**43.1**	**97.6**

66. Open-Ended Write a real-world problem that you can answer using Newton's Law of Cooling. Then answer it. **Check students' work.**

Test Prep

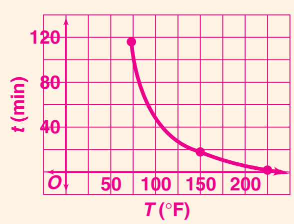
Multiple Choice

67. Which expression is equal to 3 ln 4 − 5 ln 2? **C**

A. ln (−18) **B.** $\ln\left(\frac{6}{5}\right)$ **C.** ln 2 **D.** ln 32

65a. about 43 min

b. $t = \frac{-1}{0.041}\ln\left(\frac{T-72}{164}\right)$



68. What is the value of x if $17e^{4x} = 85$? H

F. $\frac{5}{4}$ G. $\frac{\ln 85}{17 \cdot \ln 4}$ H. $\frac{\ln 5}{4}$ J. $\frac{\ln 85 - \ln 17}{\ln 4}$

69. An investment of $750 will be worth $1500 after 12 years of continuous compounding at a fixed interest rate. What is that interest rate? B

A. 2.00% B. 5.78% C. 6.93% D. 200%

Extended Response

70. The table shows the values of an investment after the given number of years of continuously compounded interest.

Years	0	1	2	3	4
Value	$500.00	$541.64	$586.76	$635.62	$688.56

a. What is the rate of interest? a–c. See margin.
b. Write an equation to model the growth of the investment.
c. To the nearest year, when will the investment be worth $1800?

Mixed Review

GO for Help

Lesson 8-5 **Solve each equation.**

71. $3^{2x} = 6561$ 4 72. $7^x - 2 = 252$ 2.846 73. $25^{2x+1} = 144$ 0.272

74. $\log 3x = 4$ $3333.\overline{3}$ 75. $\log 5x + 3 = 3.7$ 1.002 76. $\log 9 - \log x + 1 = 6$ 9.0×10^{-5}

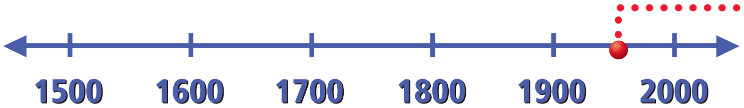
Lesson 7-7 **Find the inverse of each function. Is the inverse a function?** 77–79. See margin.

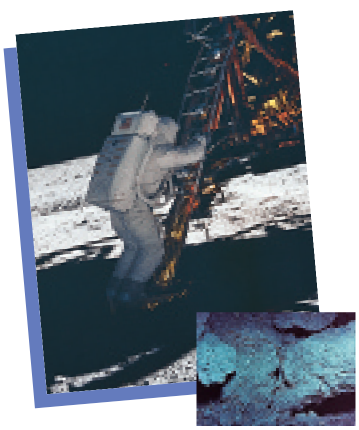
77. $y = 5x + 7$ 78. $y = 2x^3 + 10$ 79. $y = -x^2 + 5$

Lesson 6-7

80. The Nut Shop carries 30 different types of nuts. The shop special is the Triple Play, a made-to-order mixture of any three different types of nuts. How many different Triple Plays are possible? 4060 possible combinations

A Point in Time





The first manned moon landing on July 20, 1969, gave scientists a unique opportunity to test their theories about the moon's geologic history.

A logarithmic function was used to date lunar rocks. Radioactive rubidium-87 decays into stable strontium-87 at a fixed rate. The ratio r of the two isotopes in a sample can be measured and used in the equation $T = -h\frac{\ln(r + 1)}{\ln 0.5}$, where T is the age in years and h is the half-life of rubidium-87, 4.7×10^{10} years. For the lunar sample, r was measured at 0.0588, giving an approximate age of 3.87 billion years.

Go Online PHSchool.com **For:** Information about space exploration **Web Code:** age-2032

Test Prep

Resources

For additional practice with a variety of test item formats:

- Standardized Test Prep, p. 483
- Test-Taking Strategies, p. 478
- Test-Taking Strategies with Transparencies

70.[4] a. 8%

b. $A = Pe^{rt} = 500e^{0.08t}$

c. $1800 = 500e^{0.08t}$
$3.6 = e^{0.08t}$
$\ln 3.6 = 0.08t$
$\frac{\ln 3.6}{0.08} = t$
$16 \approx t$
about 16 years

[3] correct model, computation error in (b) or (c)

[2] incorrect model, solved correctly

[1] correct model, but without work shown in (c)

77. $y = \frac{x-7}{5}$; yes

78. $y = \sqrt[3]{\frac{x-10}{2}}$; yes

79. $y = \pm\sqrt{5 - x}$; no